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# On a connection governing parallel transport along 2×2 density matrices

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We investigate a connection governing parallel transport along mixed states recently defined by Uhlmann for the case of 2×2 matrices. We discuss the underlying bundle structure including singular orbits, show an interesting relation to instantons and prove that the connection fulfils the source-free Yang–Mills equation with respect to the Riemannian metric on the space of density matrices induced by the Bures metric.

*Keywords:* *connections, density matrices, Yang–Mills equation, Riemannian metric*
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## 1. Introduction

Recently a lot of work has been done on Berry's phase [1] and its nonabelian generalizations. A large list of references, probably exhausting the literature written on this subject until 1990, can be found in ref. [2]. In particular, Uhlmann proposed and discussed in a series of papers [3] a connection governing parallel transport along mixed states, which is naturally related to the concept of purification of density matrices.

Let $H$ be the Hilbert space of a quantum system with scalar product $\langle\cdot,\cdot\rangle$. A density operator is a positive trace class operator with trace one. It defines a mixed state, that means, a functional on the algebra $\mathscr{A}$ of observables via

$$\mathscr{A} \ni a \mapsto \rho(a) = \mathrm{Tr}(a\rho) . \tag{1.1}$$

The concept of purification consists in representing mixed states by pure states, that means, vectors in an extended Hilbert space $H^{\mathrm{ext}}$. This procedure is, of course, not unique. One option is to take

$$H^{\text{ext}} := H^* \otimes H\,. \tag{1.2}$$

A purification $\Phi \in H^{\text{ext}}$ of $\rho$ is then defined by the equation

$$\text{Tr}(a\rho) = \langle (\mathbf{1} \otimes a)\Phi, \Phi \rangle_{H^{\text{ext}}}\,, \tag{1.3}$$

with $\langle \cdot, \cdot \rangle_{H^{\text{ext}}}$ being the scalar product on $H^{\text{ext}}$ induced by $\langle \cdot, \cdot \rangle$. If $H \cong \mathbb{C}^n$, then

$$H^* \otimes H \cong \text{gl}(n, \mathbb{C})\,, \tag{1.4}$$

and—as one easily shows—the induced scalar product $\langle \cdot, \cdot \rangle_{H^{\text{ext}}}$ coincides in this case with the natural Hermitean sesquilinear form on $\text{gl}(n, \mathbb{C})$

$$(X, Y) \mapsto \text{Tr}(XY^*)\,. \tag{1.5}$$

As we shall see, the connection proposed by Uhlmann is directly related to this sesquilinear form.

It is the aim of this paper to investigate Uhlmann's connection for the case of $2 \times 2$ density matrices in some detail. First we make some remarks for arbitrary $n$ (section 2). In particular, we show that Uhlmann's concept of parallel transport includes the parallel transport along pure, $k$-fold degenerate quantum states, which was extensively discussed in ref. [4]. Then we discuss the bundle picture for $n=2$ including the boundary of pure states (section 3). In section 4 we show a relation of Uhlmann's connection to the canonical connection in the quaternionic Hopf bundle (instanton). In section 5 we discuss the Riemannian metric on the space of density matrices induced by the Bures metric and show that Uhlmann's connection fulfils the source-free Yang–Mills equation.

## 2. General remarks

We denote the space of complex $n \times n$ matrices by $M_n$ and by $\mathscr{D}_n$ the subspace of (not normalized) density matrices. Consider the stratification

$$M_n \ni w \mapsto \rho := ww^* \in \mathscr{D}_n\,, \tag{2.1}$$

which is in fact the orbit mapping of the right $\text{U}(n)$ action on $M_n$. This stratification is a union of fibre bundles,

$$M_n = \bigcup_k M_n(k) \to \bigcup_k \mathscr{D}_n(k) = \mathscr{D}_n\,, \tag{2.2}$$

where $M_n(k)$ [$\mathscr{D}_n(k)$] denotes the manifold of rank-$k$ matrices [rank-$k$ density matrices]. The generic (dense) stratum defined by $\det w \neq 0$ is isomorphic to the trivial principal bundle

$$\text{GL}(n, \mathbb{C}) \to \text{GL}(n, \mathbb{C})/\text{U}(n)\,, \tag{2.3}$$

and

$$M_n(k) \to \mathscr{D}_n(k) \tag{2.4}$$

are the singular strata, for $k < n$.

We denote by $P_n(k)$ $[S_n(k)]$ the space of $k$-frames [orthonormal $k$-frames] of $\mathbb{C}^n$. Moreover, let $\mathbb{C}^n \cong \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ be a fixed orthogonal decomposition and $p: \mathbb{C}^n \to \mathbb{C}^k$ the corresponding projection. In the following we consider a $k$-frame $f$ as an injective mapping $f: \mathbb{C}^k \to \mathbb{C}^n$. We denote the matrices corresponding to $p$ and $f$ by the same letters. Then the group $\mathrm{U}(k)$ acts in a natural way to the right on $P_n(k)$ and $f \mapsto ff^*$ defines the principal bundle $P_n(k) \to \mathscr{D}_n(k)$.

**Proposition 2.1.** *The mapping*

$$\Phi: P_n(k) \times_{\mathrm{U}(k)} \mathrm{U}(n-k) \backslash \mathrm{U}(n) \to M_n(k) \tag{2.5}$$

*given by*

$$\Phi([(f, [u])]) := fpu \tag{2.6}$$

*is a bundle isomorphism.*

*Proof.* Obviously, $\Phi$ is a well-defined mapping. Moreover, since $pp^* = \mathrm{Id}_{\mathbb{C}^k}$, we have $fpu(fpu)^* = ff^*$, and, therefore, $\Phi$ maps fibres into fibres. It is easy to see that $\Phi$ is one-to-one on fibres. □

**Remark 2.2.** The isomorphism (2.5) is a special case of the more general situation [5], when a group $G$ acts on a space $M$ to the right with only one orbit type, say $(H)$. In this case $M$ is isomorphic to the associated bundle $P \times_{N/H} H \backslash G$, where $N$ is the normalizer of $H$ in $G$ and $P$ is the $N/H$ principal bundle of elements of $M$ with stabilizer $H$.

**Remark 2.3.** The mapping $i: P_n(k) \to M_n(k)$ defined by

$$i(f) := fp \tag{2.7}$$

is an injective bundle homomorphism. Since $S_n(k)$ is a subbundle of $P_n(k)$, with the embedding defined by

$$f^* f = \mathrm{Id}_{\mathbb{C}^k}, \tag{2.8}$$

we conclude that $S_n(k)$ is a subbundle of $M_n(k)$.

It was shown in ref. [3] that on the above defined principal bundle, see (2.3), the equation

$$w^* \, \mathrm{d}w - \mathrm{d}w^* w = w^* w A + A w^* w \tag{2.9}$$

defines a connection form $A$. We shall give a geometrical characterization of this connection. For that purpose, let us denote the Riemannian metric on the bundle space $GL(n, \mathbb{C})$ induced from $M_n$ by $h$,

$$h(X, Y) = \mathrm{Re}\,\mathrm{Tr}(XY^*) , \tag{2.10}$$

and the corresponding vector space norm by $\|\cdot\|$,

$$\|X\|^2 = \mathrm{Tr}(XX^*) . \tag{2.11}$$

**Proposition 2.4.** *Let $X \in T_w GL(n, \mathbb{C})$. Then the following conditions are equivalent*
*(1) $X$ is horizontal with respect to $A$, $A(X) = 0$.*
*(2) $X$ fulfils*

$$w^*X - X^*w = 0 . \tag{2.12}$$

*(3) $X$ is orthogonal to the vertical subspace in the sense of $h$.*
*(4) $X$ has minimal length in the sense of $h$ among all vectors with the same projection to the base space.*

*Proof.*
(1) The equivalence of the first two points is obvious by (2.9).
(2) A vertical vector is of the form

$$V = wa , \tag{2.13}$$

with $a = -a^*$. $X$ is orthogonal to the vertical subspace iff

$$\mathrm{Tr}(a(X^*w - w^*X)) = 0 , \tag{2.14}$$

for all $a \in u(n)$, that means, iff $X^*w - w^*X = 0$. This shows the equivalence of points 2 and 3.
(3) Decompose $X = Y + V$, where $V$ is vertical and $Y$ is orthogonal to $V$. Then we have $\|X\|^2 = \|Y + V\|^2 = \|Y\|^2 + \|V\|^2$, showing the equivalence of points 3 and 4. □

In ref. [6] we have calculated $A$ and its curvature $F$ for the case $n = 2$ explicitly. The result is

$$A = \tfrac{1}{2}(\theta - \theta^*) + \frac{1}{2\,\mathrm{Tr}(w^*w)} [w^*w, \theta + \theta^*] , \tag{2.15}$$

$$F = \frac{|\det w|^2}{2[\mathrm{Tr}(w^*w)]^2} [\theta + \theta^*, \theta + \theta^*] , \tag{2.16}$$

with $\theta = w^{-1}\,\mathrm{d}w$ denoting the structure form on $GL(n, \mathbb{C})$.

For later purposes, we define the following trivial $SU(n)$ subbundle of the $U(n)$ bundle (2.3):

$$Q_n := \{w \in \mathrm{GL}(n, \mathbb{C}) : \det w \in \mathbb{R}_+\} . \tag{2.17}$$

**Proposition 2.5.** *The connection $A$ defined by (2.9) is reducible to $Q_n$.*

*Proof.* We have to show that on $Q_n$ $A$ takes values in the subalgebra $\mathrm{su}(n)$, that means, $\mathrm{Tr}\, A = 0$. Since $\det w \neq 0$, we can multiply the defining equation (2.9) by $(w^* w)^{-1}$ to the left. This yields

$$w^{-1}\, \mathrm{d}w - w^{-1} w^{*-1}\, \mathrm{d}w^* w = A + (w^* w)^{-1} A w^* w , \tag{2.18}$$

and—after taking the trace—we have

$$\mathrm{Tr}\, A = \tfrac{1}{2} \mathrm{Tr}(w^{-1}\, \mathrm{d}w - (w^{-1}\, \mathrm{d}w)^*) . \tag{2.19}$$

From

$$\mathrm{d}(\det w) = \det w \cdot \mathrm{Tr}(w^{-1}\, \mathrm{d}w) \tag{2.20}$$

we get that $\mathrm{Tr}(w^{-1}\, \mathrm{d}w)$ is a real-valued form on $Q_n$ and, therefore, $\mathrm{Tr}\, A = 0$ on $Q_n$. □

It is interesting to discuss the mathematical meaning of the defining equation (2.9) for the case of singular matrices. This will be done in another paper [7]. In particular, we will show

**Proposition 2.6.** *The distribution*

$$F_w := \{X \in \mathrm{T}_w M_n(k) : w^* X - X^* w = 0\} \tag{2.21}$$

*defines a connection on the bundle $M_n(k) \to \mathcal{D}_n(k)$.* □

**Remark 2.7.** Due to remark 2.2, the Stiefel bundle $S_n(k)$ is a subbundle of $M_n(k)$. Since $S_n(k)$ is the purification space of $k$-fold degenerate quantum states, it is interesting to ask which connection on $S_n(k)$ is induced by the above defined connection. From eqs. (2.7) and (2.8) we see that every vector $X \in \mathrm{T}_s i(S_n(k))$ fulfils

$$X^* s + s^* X = 0 , \tag{2.22}$$

for every $s \in i(S_n(k))$. Inserting this into the horizontality condition (2.21) yields that $X$ is horizontal iff

$$s^* X = 0 . \tag{2.23}$$

The corresponding connection form is

$$A = s^*\, \mathrm{d}s . \tag{2.24}$$

Its pull-back under the embedding $i$, see eq. (2.7), to the Stiefel bundle gives the canonical connection—as already observed in refs. [3,8]. This connection was used in ref. [4] for the discussion of $k$-fold degenerate quantum states.

## 3. Structure of singular orbits and connection on the boundary for the case $n=2$

For $n=2$ we have the stratification

$$M_2 \ni w \mapsto \rho = ww^* \in \mathscr{D}_2 , \tag{3.1}$$

which has only one nontrivial singular orbit type, defined by rank $w=1$. From now on we restrict ourselves to normalized density matrices,

$$\mathrm{Tr}\,\rho \equiv \mathrm{Tr}(ww^*) = 1 . \tag{3.2}$$

We denote the space of matrices $w$ satisfying (3.2) by $\hat{M}_2$, the subspaces of matrices of rank $k$ by $\hat{M}_2(k)$, $k=1, 2$, and—correspondingly—the space of normalized density matrices by $\hat{\mathscr{D}}_2$, the subspaces of rank $k$ matrices by $\hat{\mathscr{D}}_2(k)$, $k=1, 2$.

We see from eq. (3.2) that

$$\hat{M}_2 \cong \mathrm{S}^7 . \tag{3.3}$$

Moreover, every $\rho \in \hat{\mathscr{D}}_2$ can be represented as

$$\rho = \tfrac{1}{2}\cdot \mathbf{1} + x_k \sigma^k , \quad \sum_{k=1}^{3} x_k^2 \leq \tfrac{1}{4} , \tag{3.4}$$

$x_k \in \mathbb{R}^3$, $(\sigma^k)$ the Pauli matrices. It follows that

$$\det \rho = \tfrac{1}{4} - \sum_{k=1}^{3} x_k^2 , \tag{3.5}$$

and we have

$$\hat{\mathscr{D}}_2 \cong \mathrm{D}^3 , \tag{3.6}$$

$$\hat{\mathscr{D}}_2(1) \cong \partial \mathrm{D}^3 \cong \mathrm{S}^2 , \tag{3.7}$$

$$\hat{\mathscr{D}}_2(2) \cong \mathrm{Int}(\mathrm{D}^3) , \tag{3.8}$$

where $\mathrm{D}^3$ denotes a three-dimensional disc. Putting $x_0^2 = \det \rho$ we can identify $\hat{\mathscr{D}}_2$ with the upper half shell of a three-sphere and the boundary $\hat{\mathscr{D}}_2(1)$ of pure states with its equator. Obviously, the generic stratum

$$\hat{M}_2(2) \cong \mathrm{S}^7 \to \mathrm{Int}(\mathrm{D}^3) \cong \hat{\mathscr{D}}_2(2) \tag{3.9}$$

is a principal U(2) bundle. The structure of the bundle

$$\hat{M}_2(1) \to \hat{\mathscr{D}}_2(1) \tag{3.10}$$

follows from proposition 2.1. We have

$$M_2(1) \cong P_2(1) \times_{\mathrm{U}(1)} \mathrm{U}(1) \backslash \mathrm{U}(2) . \tag{3.11}$$

After inserting (2.6) into (3.2) and comparing with (2.8) we see that

$$\hat{M}_2(1) \cong S_2(1) \times_{\mathrm{U}(1)} U(1) \backslash U(2) . \tag{3.12}$$

Finally, $\mathrm{U}(1) \backslash \mathrm{U}(2) \cong \mathrm{S}^3$ and the Stiefel bundle $S_2(1) \to \hat{\mathscr{D}}_2(1)$ coincides with the complex Hopf bundle $\mathrm{S}^3 \to \mathrm{S}^2$; therefore,

$$\hat{M}_2(1) \cong \mathrm{S}^3 \times_{\mathrm{U}(1)} \mathrm{S}^3 . \tag{3.13}$$

Due to remark 2.3 we have an embedding of the complex Hopf bundle into $\hat{M}_2(1)$ and due to remark 2.7, the induced connection on the boundary of pure states coincides with the canonical Hopf bundle connection (gauge potential of a magnetic monopole).

An explicit description in terms of homogeneous coordinates is obtained as follows: According to remark 2.3 and (3.12) the embedding of the Stiefel bundle $S_2(1)$ into $\hat{M}_2(1)$ is given by

$$\mathrm{S}^3 \cong S_2(1) \ni \begin{pmatrix} a \\ b \end{pmatrix} \mapsto s := \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in \hat{M}_2(1) , \tag{3.14}$$

where $|a|^2 + |b|^2 = 1$, and every $w \in \hat{M}_2(1)$ takes the form

$$w = su = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} u , \tag{3.15}$$

with $u$ being a representative of the class $[u] \in \mathrm{U}(1) \backslash \mathrm{U}(2) \cong \mathrm{S}^3$. The embedding of the U(1) factor is given by

$$U(1) \ni \mathrm{e}^{\mathrm{i}\alpha} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha} \end{pmatrix} \in \mathrm{U}(2) .$$

Of course (3.15) is unique up to a $N(\mathrm{U}(1))/\mathrm{U}(1) \cong \mathrm{U}(1)$ factor, $w = shh^{-1}u$, and this factor may be represented as

$$h = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & 1 \end{pmatrix} .$$

The bundle projection in these coordinates is now

$$\mathbb{C}^2 \supset \mathrm{S}^3 \ni \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \left[ \begin{pmatrix} a\mathrm{e}^{\mathrm{i}\alpha} \\ b\mathrm{e}^{\mathrm{i}\alpha} \end{pmatrix} \right] \in \mathbb{C}\mathrm{P}^1 , \tag{3.16}$$

making the Hopf bundle structure explicit. Finally, the pull-back of the connection form (2.24) takes the form

$$A = \bar{a}\, \mathrm{d}a + \bar{b}\, \mathrm{d}b , \tag{3.17}$$

which is the canonical Hopf bundle connection in homogeneous coordinates.

## 4. A relation to instantons

The fact that the connection defined by Uhlmann coincides for pure states with a canonical geometric structure suggests that for mixed states something similar happens. This is indeed the case. Putting

$$w=(1/\sqrt{2})(x+\mathrm{i}z)\ , \tag{4.1}$$

with $(x, z)$ being a pair of quaternions, see ref. [9], we identify $S^7$ with the bundle space of the quaternionic Hopf bundle over $\mathbb{HP}^1 \cong S^4$. The subbundle $\hat{Q}_2 = Q_2 \cap S^7$, with $Q_2$ defined by eq. (2.17), is in quaternionic coordinates given by

$$\mathrm{Tr}(xz^*)=0\ . \tag{4.2}$$

In the quaternionic Hopf bundle we have the canonical connection (instanton)

$$\mathring{A}=x^*\,\mathrm{d}x+z^*\,\mathrm{d}z\ , \tag{4.3}$$

see ref. [9].

**Proposition 4.1.** *The connection form A (treated as a connection on $\hat{Q}_2$) coincides with the restriction of $\mathring{A}$ to $\hat{Q}_2$.*

*Proof.* From the defining equation (2.9) we have

$$\mathrm{Tr}(Aw^*w)=\tfrac{1}{2}\mathrm{Tr}(w^*\,\mathrm{d}w-\mathrm{d}w^*w)\ . \tag{4.4}$$

On the other hand, a simple calculation shows that

$$\begin{aligned} &w^*wA+Aw^*w \\ &\quad=\mathrm{Tr}(w^*w)\,A+w^*w\,\mathrm{Tr}\,A+[\mathrm{Tr}(Aw^*w)-\mathrm{Tr}(w^*w)\,\mathrm{Tr}\,A]\mathbf{1} \\ &\quad=A+\mathrm{Tr}(Aw^*w)\,\mathbf{1}+\mathrm{Tr}\,A(w^*w-\mathbf{1})\ . \end{aligned} \tag{4.5}$$

Inserting (4.4) and (4.5) into the defining equation (2.9) we get

$$A=w^*\,\mathrm{d}w-\mathrm{d}w^*w-\tfrac{1}{2}\mathrm{Tr}(w^*\,\mathrm{d}w-\mathrm{d}w^*w)-\mathrm{Tr}\,A(w^*w-\mathbf{1})\ . \tag{4.6}$$

Using (4.1) yields

$$w^*\,\mathrm{d}w-\mathrm{d}w^*w \tag{4.7}$$

$$=\tfrac{1}{2}(x^*\,\mathrm{d}x-\mathrm{d}x^*x+z^*\,\mathrm{d}z-\mathrm{d}z^*z)+\tfrac{1}{2}\mathrm{i}(x^*\,\mathrm{d}z-\mathrm{d}x^*z+\mathrm{d}z^*x-\mathrm{d}x^*z)$$

$$=\mathring{A}+\tfrac{1}{2}\mathrm{i}(x^*\,\mathrm{d}z-\mathrm{d}x^*z+\mathrm{d}z^*x-\mathrm{d}x^*z)\ . \tag{4.8}$$

A simple quaternionic calculation shows that the quantity $\tau \equiv x^* \mathrm{d}z - \mathrm{d}x^* z + \mathrm{d}z^* x - \mathrm{d}x^* z$ fulfils

$$\tau = \tfrac{1}{2} \mathrm{Tr}\, \tau \cdot \mathbf{1} \,. \tag{4.9}$$

Therefore, we obtain

$$w^* \,\mathrm{d}w - \mathrm{d}w^* w - \tfrac{1}{2} \mathrm{Tr}(w^* \,\mathrm{d}w - \mathrm{d}w^* w) = \mathring{A} \,. \tag{4.10}$$

Taking into account that on $\hat{Q}_2$ we have $\mathrm{Tr}\, A = 0$, see proposition 2.5, we see from (4.6) and (4.10) that $A$ and $\mathring{A}$ coincide on $\hat{Q}_2$. □

## 5. Field equations

### 5.1. THE NATURAL RIEMANNIAN METRIC ON $\mathscr{D}_2(2)$

The norm (2.11) induces a (topological) metric on the space of density matrices $\mathscr{D}_2$, called the Bures metric [10],

$$d_\mathrm{B}(\rho, \mu) := \inf \|w - v\| \,, \tag{5.1}$$

with $\rho = ww^*$, $\mu = vv^*$. Obviously,

$$\|w - v\|^2 = \mathrm{Tr}((w - v)(w - v)^*) = 2 - 2 \,\mathrm{Re}\, \mathrm{Tr}(wv^*) \,,$$

and, therefore,

$$d_\mathrm{B}(\rho, \mu) = \sqrt{2 - 2 \sup\{\mathrm{Re}\, \mathrm{Tr}(wv^*)\}} \,. \tag{5.2}$$

The quantity

$$t(\rho, \mu) := \sup\{\mathrm{Re}\, \mathrm{Tr}(wv^*)\} \tag{5.3}$$

is the transition probability between mixed states $\rho$ and $\mu$ [3]. One finds [11]

$$d_\mathrm{B}(\rho, \mu) = \sqrt{2 - 2\, \mathrm{Tr}(\mu^{1/2} \rho \mu^{1/2})^{1/2}} \,. \tag{5.4}$$

Moreover, we have a natural Riemannian metric $g$ on the manifold of nonsingular density matrices,

$$g(X, Y) := \mathrm{Re}\, \mathrm{Tr}(X^\mathrm{h} (Y^\mathrm{h})^*) \,, \tag{5.5}$$

where $X$, $Y \in \mathrm{T}\mathscr{D}_n(n)$ and $X^\mathrm{h}$ and $Y^\mathrm{h}$ are horizontal lifts in the sense of the given connection. Obviously, $g$ is given by

$$g = s^* \{\mathrm{Re}\, \mathrm{Tr}((\mathrm{d}w - wA)(\mathrm{d}w - wA)^*)\} \,, \tag{5.6}$$

where $s$ is an arbitrary section of the principal bundle (2.3) and $\mathrm{d}w - wA$ is the horizontal component of $\mathrm{d}w$.

**Proposition 5.1.** *For $n = 2$ we get*

$$g=\frac{1}{2}\mathrm{Tr}(\mathrm{d}\rho\cdot\mathrm{d}\rho)+\mathrm{d}(\det\rho)^{1/2}\cdot\mathrm{d}(\det\rho)^{1/2} . \tag{5.7}$$

*Proof.* From (2.15) we have

$$\mathrm{d}w-wA=\frac{1}{2}w(\theta+\theta^*-[w^*w,\theta+\theta^*]) . \tag{5.8}$$

Using the (unitary gauge) section

$$s(\rho):=\rho^{1/2} , \tag{5.9}$$

we get

$$\begin{aligned} s^*(\mathrm{d}w-wA) &= \tfrac{1}{2}\rho^{1/2}(\rho^{-1/2}\,\mathrm{d}\rho^{1/2}+\mathrm{d}\rho^{1/2}\,\rho^{-1/2}-[\rho,\rho^{-1/2}\,\mathrm{d}\rho^{1/2}+\mathrm{d}\rho^{1/2}\,\rho^{-1/2}]) \\ &= \tfrac{1}{2}\rho^{1/2}(\rho^{-1/2}\,\mathrm{d}\rho\,\rho^{-1/2}-[\rho,\rho^{-1/2}\,\mathrm{d}\rho\,\rho^{-1/2}]) \\ &= \tfrac{1}{2}(\mathrm{d}\rho\,\rho^{-1/2}-\rho\,\mathrm{d}\rho\,\rho^{-1/2}+\mathrm{d}\rho\,\rho^{1/2}) . \end{aligned} \tag{5.10}$$

Inserting (5.10) into (5.6) yields

$$g=\frac{1}{4}\mathrm{Tr}\{(\rho^{-1}+2-\rho)(\mathrm{d}\rho)^2+(\rho-2)\,\mathrm{d}\rho\,\rho^{-1}\,\mathrm{d}\rho\,\rho\} . \tag{5.11}$$

Denoting

$$\lambda:=\det\rho^{1/2} , \tag{5.12}$$

the Cayley-Hamilton theorem for $\rho$ yields

$$\rho^2-\rho+\lambda^2\cdot\mathbf{1}=0 . \tag{5.13}$$

As a consequence we get

$$\rho^{-1}=\frac{1}{\lambda^2}(\mathbf{1}-\rho) . \tag{5.14}$$

Inserting (5.13) and (5.14) into (5.11), we get

$$g=\frac{1}{4}\mathrm{Tr}\left\{\left(1+\frac{1}{\lambda^2}\right)(\mathrm{d}\rho)^2-\frac{2}{\lambda^2}\rho(\mathrm{d}\rho)^2+\frac{1}{\lambda^2}\rho\,\mathrm{d}\rho\,\rho\,\mathrm{d}\rho\right\} . \tag{5.15}$$

Finally, from (5.13) one easily derives the following identities:

$$\mathrm{Tr}(\rho(\mathrm{d}\rho)^2)=\frac{1}{2}\mathrm{Tr}(\mathrm{d}\rho)^2 , \tag{5.16}$$

$$\mathrm{Tr}(\rho\,\mathrm{d}\rho\,\rho\,\mathrm{d}\rho)=\lambda^2[\mathrm{Tr}(\mathrm{d}\rho)^2+4(\mathrm{d}\lambda)^2] . \tag{5.17}$$

Inserting (5.16) and (5.17) into (5.15) gives (5.7). □

On the other hand, a Riemannian metric can be obtained by taking the Hessian of the square of the distance function defined by the Bures metric,

$$\tilde{g}_\rho = \tfrac{1}{2}\ \mathrm{Hess}_\rho\, d_{\mathrm{B}}^2(\rho, \cdot)\ . \tag{5.18}$$

Applying this formula to the $n=2$ case, one finds that $\tilde{g}$ coincides with $g$ given by (5.7). Using other techniques, formula (5.7) was independently obtained in ref. [12]. In coordinates $(x_0, x_k)$, introduced in section 3, we get

$$g = \sum_{k=1}^{3} \mathrm{d}x_k^2 + \mathrm{d}x_0^2\ , \tag{5.19}$$

showing that $g$ coincides with the Riemannian metric on the upper half shell of $S^3$ of radius $\frac{1}{2}$. In these coordinates the corresponding volume form is given by

$$v = \frac{1}{2x_0}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\ . \tag{5.20}$$

### 5.2. THE YANG–MILLS EQUATION FOR THE CASE $n=2$

For the case of normalized density matrices formulae (2.15) and (2.16) for connection and curvature take the form

$$A = \tfrac{1}{2}(\theta - \theta^*) + \tfrac{1}{2}[w^*w, \theta + \theta^*]\ , \tag{5.21}$$

$$F = \tfrac{1}{2}|\det w|^2[\theta+\theta^*, \theta+\theta^*] + \tfrac{1}{2}\mathrm{Tr}\{w^*w(\theta+\theta^*)\}[w^*w, \theta+\theta]\ . \tag{5.22}$$

Choosing section (5.9) we get the following pull-backs (gauge potential and its field strength):

$$A = [\rho^{1/2}, \mathrm{d}\rho^{1/2}]\ , \tag{5.23}$$

$$F = \tfrac{1}{2}(\mathrm{Tr}\,\rho^{1/2})^2[\mathrm{d}\rho^{1/2}, \mathrm{d}\rho^{1/2}] - \mathrm{Tr}\,\rho^{1/2}\cdot\mathrm{Tr}(\mathrm{d}\rho^{1/2}) \wedge [\rho^{1/2}, \mathrm{d}\rho^{1/2}]\ . \tag{5.24}$$

(We use the same symbols $A$ and $F$ for the pull-backs!) The last formula is the result of a lengthy, but simple calculation, where one has to use the Cayley–Hamilton theorem

$$\rho - \mathrm{Tr}\,\rho^{1/2}\cdot\rho^{1/2} + \det \rho^{1/2}\cdot \mathbf{1} = 0\ , \tag{5.25}$$

and consequences obtained by differentiating it.

Further calculations will be performed in stereographic projection coordinates $(z_k)$ on $S^3$. Denoting $\|z\|^2 \equiv \sum_{k=1}^3 z_k^2$, we have

$$x_0 = \frac{1}{2}\frac{\|z\|^2 - 1}{\|z\|^2+1}\ , \qquad x_k = \frac{z_k}{\|z\|^2+1}\ . \tag{5.26}$$

Moreover, we denote by $\eta_{ij}$ the Euclidean metric on $\mathbb{R}^3$ and put $\mu \equiv \|z\|^2 + 1$. Then we get

$$g = \frac{1}{\mu^2}\sum_{k=1}^{3} \mathrm{d}z_k^2\ , \tag{5.27}$$

that means,

$$g_{ij}=\frac{1}{\mu^2}\eta_{ij}\,,\qquad g^{ij}=\mu^2\eta^{ij}\,. \tag{5.28}$$

For density matrices we have

$$\rho=\tfrac{1}{2}\cdot\mathbf{1}+\frac{z_k}{\mu}\sigma^k\,, \tag{5.29}$$

$$\rho^{1/2}=\frac{1}{\sqrt{2\mu(\mu-1)}}\left[(\mu-1)\cdot\mathbf{1}+z_k\sigma^k\right]. \tag{5.30}$$

Inserting this into (5.23) and (5.24), we obtain

$$A=\frac{\mathrm{i}}{\mu(\mu-1)}\epsilon_{lkm}z^l\,\mathrm{d}z^k\,\sigma^m\,, \tag{5.31}$$

$$F=\frac{\mathrm{i}}{\mu^2(\mu-1)^2}\{\epsilon_{ij}{}^l(\mu(\mu-1)\eta_{lm}-z_lz_m)$$
$$+(2\mu-1)z^l(\epsilon_{lim}z_j-\epsilon_{ljm}z_i)\}\sigma^m\,\mathrm{d}z^i\wedge\mathrm{d}z^j\,. \tag{5.32}$$

**Proposition 5.2.** *The above defined gauge field fulfils the source-free Yang–Mills equation*·

$$D{*}F=0\,. \tag{5.33}$$

*Proof.* The canonical volume form (5.20) takes the form

$$v=\frac{1}{6\mu^3}\epsilon_{klm}\,\mathrm{d}z^k\wedge\mathrm{d}z^l\wedge\mathrm{d}z^m\,.$$

Using this and (5.32), we get for the Hodge dual of $F$ the following formula:

$${*}F=\frac{2\mathrm{i}}{\mu(\mu-1)}\left[-(\mu-1)\eta_{kl}+2z_kz_l\right]\sigma^k\,\mathrm{d}z^l\,. \tag{5.34}$$

Now one has to calculate

$$D{*}F=\tfrac{1}{2}({*}F_{l,k}-{*}F_{k,l}+[A_k,{*}F_l]-[A_l,{*}F_k])\,\mathrm{d}z^k\wedge\mathrm{d}z^l\,. \tag{5.35}$$

Using (5.31) and (5.34) we obtain

$${*}F_{l,k}=\frac{4\mathrm{i}}{\mu(\mu-1)}\left\{\eta_{mk}z_l+\eta_{ml}z_k+\eta_{kl}z_m-\frac{2(2\mu-1)}{\mu(\mu-1)}z_mz_kz_l-\frac{1}{\mu}\eta_{ml}z_k\right\}\sigma^m, \tag{5.36}$$

$$[A_k, *F_l] = \frac{4i}{\mu^2(\mu-1)} \left\{ -\eta_{mk} z_l - \eta_{kl} z_m + \frac{2}{(\mu-1)} z_m z_k z_l \right\}. \quad (5.37)$$

Combining these two formulae we get

$$*F_{l,k} - *F_{k,l} = \frac{4i}{\mu^2(\mu-1)} (\eta_{mk} z_l - \eta_{ml} z_k) \sigma^m$$
$$= -([A_k, *F_l] - [A_l, *F_k]) . \quad (5.38)$$

□

Finally, taking into account remark 2.7, we notice that the U(1) connection on the boundary of pure states fulfils the source-free Maxwell equation.

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